Semiclassical quantization of hyperbolic map on torus

2004 J. Phys. A: Math. Gen. 37 8007

(http://iopscience.iop.org/0305-4470/37/33/003)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 8007–8016 PII: S0305-4470(04)80042-8

Semiclassical quantization of hyperbolic map on torus

Mitsusada M Sano

Graduate School of Human and Environmental Studies, Kyoto University, Sakyo, Kyoto 606-8501, Japan

Received 30 April 2004, in final form 30 April 2004
Published 4 August 2004
Online at stacks.iop.org/JPhysA/37/8007
doi:10.1088/0305-4470/37/33/003

Abstract
We consider the semiclassical quantization of two hyperbolic maps on torus, i.e., the baker's map and the sawtooth map. We demonstrate that for both maps, the semiclassical quantization scheme based on the Riemann–Siegel lookalike formula fails to yield its eigenvalues. The reason for this failure is that the truncation of the infinite series w.r.t. the periodic orbits cannot work for the quantized hyperbolic map on a torus. We show that for the baker's map, an alternative semiclassical quantization scheme including all periodic orbit contribution yields its eigenvalues with reasonable accuracy, although they are, in general, complex-valued. The same scheme for the sawtooth map is constructed.

PACS number: 05.45.Mt

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The semiclassical quantization of a chaotic system is one of the challenging problems in physics and chemistry. For hyperbolic systems, the Gutzwiller trace formula is a powerful tool (Gutzwiller 1971, 1990). During the last two decades, many numerical studies of testing the Gutzwiller trace formula have surely proved its validity for the semiclassical quantization. However, there are some difficulties in using the Gutzwiller trace formula: (1) the proliferation of the number of periodic orbits when the period is increased. The number of periodic orbits is, in fact, infinite. (2) How do we find the periodic orbits for a given system? The first point is overcome by using the analogy with the Riemann zeta function. Berry and Keating rewrote the Gutzwiller trace formula into the zeta function (sometimes called the Guzwiller–Voros zeta function), and applied the Riemann–Siegel formula for the Riemann zeta function to the Gutzwiller–Voros zeta function (Berry and Keating 1990, 1992, Keating 1992). The derived zeta function is called the Riemann–Siegel lookalike formula. The zeros of this zeta function are the eigenvalues (eigenangles) which we want. This formula has the great advantage that it reduces the infinite sum to the finite sum w.r.t. periodic orbits, i.e., analytical bootstrap. Berry and Keating's method was immediately applied to a quantized map system (Smilansky 1993). We call this method of Smilansky 'method A'.

The author applied method A to the sawtooth map (Sano 1996). However, method A did not work for the sawtooth map with some parameter values. Precisely speaking, the derived zeta function, i.e., the Riemann–Siegel lookalike formula, does not cross the zero axis near some quantum exact eigenvalues. To date, the reason for this failure has remained unknown.

An alternative semiclassical evaluation of the trace was proposed for the quantized baker's map (Dittes *et al* 1994). Their method uses the weighted Perron–Frobenius operator. The semiclassical trace is exactly the trace of the weighted Perron–Frobenius operator. We call this method 'method B'. In order to elucidate why the quantized chaotic system exhibits good agreement with the prediction of the random matrix theory, method B was applied to investigate the action correlation for the baker's map (Tanner 1999, Sano 2000). Recently Smilansky and Verdene reinvestigated the action correlation of the baker's map (Smilansky and Verdene 2003).

In this paper, for the quantized baker's map, we carry out semiclassical quantization both by method A and by method B. We show that method B provides us a semiclassical quantization method summing up all periodic orbit contributions, i.e., a kind of resummation method. The numerical result for both methods shows that (a) method A fails in the semiclassical evaluation of the eigenvalues, (b) but method B gives the semiclassical eigenvalues with reasonable accuracy, although the semiclassical eigenvalues are, in general, in the complex domain and an infinite number of irrelevant semiclassical eigenvalues exists. It has already been pointed out that the semiclassical eigenangles $\{\omega_n\}_{n=1}^{N}$ are, in general, complex-valued and the error in the semiclassical eigenangles is $O(\hbar)$ for a smooth map and $O(\hbar^{1/2})$ for a discontinuous map (Keating 1994). The conclusion from the numerical result is that in order to carry out the semiclassical quantization for the map on a torus, method A is not appropriate and a resummation like method B is needed. In addition, we present a formal theory of method B for the sawtooth map, although we do not carry out a numerical study.

The organization of this paper is as follows. In section 2, we present general result on the spectral determinant for the quantized map on a torus and the result of Smilansky's Riemann–Siegel lookalike formula. In section 3, we develop the semiclassical quantization of the baker's map by method A and by method B. In section 4, the semiclassical quantization of the sawtooth map by method B is constructed. But we do not carry out a numerical study because of its difficulty. In section 5, we summarize the result.

2. Spectral determinant

Consider a quantum map on a torus. The phase space is periodic both in position space and in momentum space. This makes the Planck constant $\hbar = \mathcal{A}/(2\pi N)$, where $\mathcal{A}$ is the area of the torus and N is a positive integer. Hereafter we set $\mathcal{A} = 1$ for our convenience. The wavefunction is evolved by the Floquet operator $\hat{U}$,

$$\psi_{t+1} = \hat{U}\psi_t. \tag{1}$$

$\hat{U}$ is the $N \times N$-unitary matrix. The eigenvalue problem is now

$$\hat{U}\psi = \mathrm{e}^{\mathrm{i}\omega}\psi. \tag{2}$$

We define the set of eigenvalues $\{\omega_n\}_{n=1}^N$. The density of states is given by

$$\begin{aligned} d(\omega) &= \sum_{n=1}^{N}\sum_{l=-\infty}^{\infty} \delta(\omega-\omega_n-2\pi l) \\ &= \frac{N}{2\pi}+\frac{1}{\pi}\,\mathrm{Re}\sum_{n=1}^{\infty}\mathrm{Tr}(\hat{U}^n)\,\mathrm{e}^{-\mathrm{i}\omega n} \\ &= d_0 + d_{\mathrm{osc}}(\omega). \end{aligned} \tag{3}$$

If the map is hyperbolic, then the periodic orbits of the map are isolated. With this condition, the semiclassical approximation of the trace of $\hat{U}^n$ is given in the lowest order of $\hbar$ by

$$\mathrm{Tr}(\hat{U}^n) \simeq \mathrm{Tr}^{(\mathrm{sc})}(\hat{U}^n) = \mathrm{i}^n \sum_{n=n_p r}^{\infty} \frac{n_p}{\left|\det\left(M_p^n - I\right)\right|^{1/2}} \exp\left[\frac{\mathrm{i}}{\hbar}S_p r - \frac{\mathrm{i}\pi\mu_p r}{2}\right], \tag{4}$$

where p represents the primitive periodic orbit and n_p, M_p, S_p and μ_p are the period, the monodromy matrix, the action and the Maslov index of p, respectively. We define the spectral determinant for the Floquet operator $\hat{U}$ for the quantized map,

$$\begin{aligned} P(z) &= \det(1-z^{-1}\hat{U}) \\ &= \sum_{n=0}^{N} a_n z^{n-N}. \end{aligned} \tag{5}$$

a_n are related to each other as follows:

$$a_{N-k} = -\frac{1}{k}\sum_{n=1}^{k} a_{N-k+1}\,\mathrm{Tr}(\hat{U}^n), \tag{6}$$

with $a_N = 1$. This relation is called Newton's relation. Newton's relation is a kind of bootstrapping, namely the first $N/2$ a_n ($n = 0, 1, 2, \ldots, [N/2]-1$) are determined by the latter $N/2$ a_n ($n = [N/2], [N/2]+1, \ldots, N-1$). We further define the zeta function $Z(\omega)$.

$$Z(\omega) = C(\omega)\det(1-z^{-1}\hat{U}), \tag{7}$$

where

$$C(\omega) = \exp[-\mathrm{i}(\Theta - N\omega)/2] \tag{8}$$

and

$$\mathrm{e}^{\mathrm{i}\Theta} = \mathrm{e}^{\mathrm{i}\sum_k^N \omega_k - N\pi}\det(-\hat{U}). \tag{9}$$

Here we set $z = \mathrm{e}^{\mathrm{i}\omega}$. Using the relation $\ln\det = \mathrm{Tr}\ln$, $Z(\omega)$ can be expanded as

$$Z(\omega) = C(\omega)\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}(\hat{U}^n)z^{-n}\right]. \tag{10}$$

The density of states can be written in terms of the spectral determinant:

$$d(\omega) = \frac{1}{\pi}\,\mathrm{Re}\frac{\mathrm{d}}{\mathrm{d}\omega}\ln Z(\omega). \tag{11}$$

The Riemann–Siegel lookalike formula $Z^{(\mathrm{sc,RS})}(\omega)$ for the zeta function $Z(\omega)$ is given as follows (Smilansky 1993):

$$Z^{(\mathrm{sc,RS})}(\omega) = \sum_{n=0}^{[N/2]-\epsilon_N}\left\{A_{N-n}^*\,\mathrm{e}^{\mathrm{i}(n-\frac{N}{2})\omega} + A_{N-n}\,\mathrm{e}^{-\mathrm{i}(n-\frac{N}{2})\omega}\right\} + \frac{1}{2}\epsilon_N(A_{N/2}+A_{N/2}^*), \tag{12}$$

where

$$\epsilon_N = \begin{cases} 1 & N \text{ even} \\ 0 & N \text{ odd} \end{cases} \tag{13}$$

and

$$A_{N-k} = -\frac{1}{k}\sum_{n=1}^{k} A_{N-k+n}\,\mathrm{Tr}^{(\mathrm{sc})}(\hat{U}^n), \tag{14}$$

with $A_N = 1$. A_{N-k} are recursively determined by the semiclassical values of the trace. Only $N/2$ traces are needed for the evaluation of $Z^{(\mathrm{sc,RS})}(\omega)$, i.e., analytical bootstrapping.

3. Quantized baker's map

The classical baker map's is defined by

$$x_{t+1} = 2x_t - [2x_t], \qquad y_{t+1} = \frac{y_t + [2x_t]}{2}. \tag{15}$$

The corresponding quantum baker's map is defined as follows. The original quantum baker's map does not have symmetry which the classical map possesses (Balazs and Voros 1987, 1989). We use the quantum baker's map which has this symmetry (Saraceno 1990, Saraceno and Voros 1994). The wavefunction in the position space is mapped by the Floquet operator $\hat{U}_B$,

$$\psi_{t+1} = \hat{U}_B\psi_t. \tag{16}$$

Here the Floquet operator is defined by

$$\hat{U}_B = G_N^{-1}\begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix}, \tag{17}$$

where

$$(G_N)_{kn} = \langle k|n\rangle = \frac{1}{\sqrt{N}}\exp\left[-\frac{2\pi\mathrm{i}}{N}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right], \tag{18}$$

with $k, n = 0, 1, 2, \ldots, N-1$. The eigenvalue problem is

$$\hat{U}_B\psi = \mathrm{e}^{\mathrm{i}\omega}\psi. \tag{19}$$

The spectral determinant for the quantized baker's map is given by

$$Z_B(\omega) = C_B(\omega)\det(1 - z^{-1}\hat{U}_B), \tag{20}$$

where B stands for the baker's map. Now we define the weighted Perron–Frobenius operator $\mathcal{L}_B$ for the baker's map (Dittes *et al* 1994),

$$\mathcal{L}_B(x', x; N) = \sqrt{2}\exp\left[\frac{\mathrm{i}}{\hbar}S(x)\right]\delta(x' - (2x - [2x])), \tag{21}$$

where

$$S(x) = x[2x] - \tfrac{1}{2}([2x] + x). \tag{22}$$

It can easily be shown that

$$\mathrm{Tr}\big(\hat{U}_B^n\big) \simeq \mathrm{Tr}^{(\mathrm{sc})}\big(\hat{U}_B^n\big) = \mathrm{Tr}\big(\mathcal{L}_B^n\big). \tag{23}$$

Using equation (23), we have

$$Z_B(\omega) \simeq Z_B^{(\mathrm{sc})}(\omega) = C_B(\omega)\det(1 - z^{-1}\mathcal{L}_B). \tag{24}$$

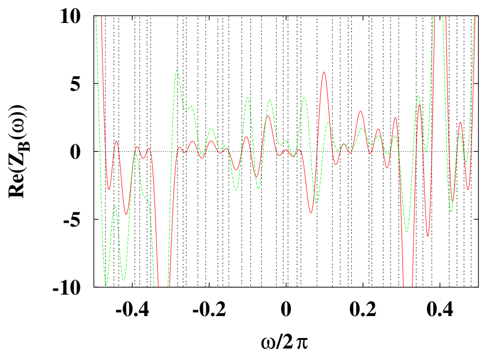

Figure 1. Semiclassical quantization of the baker's map using the Riemann–Siegel lookalike formula: $N = 1/(2\pi\hbar) = 80$. The data are for even parity. Vertical dashed lines indicate the position of the quantum exact eigenvalue. The red curve (solid curve) represents the value of the zeta function $\mathrm{Re}(Z_B(\omega))$. The green curve (dotted curve) represents the value of the zeta function $\mathrm{Re}(Z_B^{(\mathrm{sc})}(\omega))$. It is clearly seen that $\mathrm{Re}(Z_B^{(\mathrm{sc})}(\omega))$ does not cross the zero axis near some exact quantum eigenvalues.

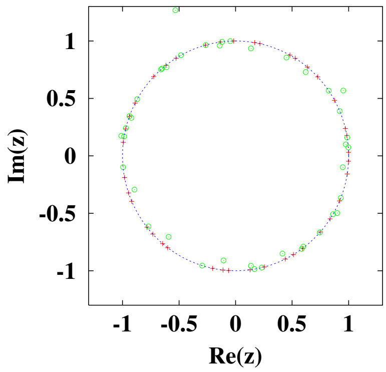

Figure 2. Semiclassical eigenvalues in the complex plane (semiclassical quantization of the baker's map using the Riemann–Siegel lookalike formula): $N = 1/(2\pi\hbar) = 80$. The data are for even parity. Circles indicates the positions of semiclassical eigenvalues. Crosses indicate the quantum exact eigenvalues. Some of the semiclassical eigenvalues are away from the unit circle. The coincidence is rather bad. In fact, one of them is out of this figure.

In figure 1, we depict the quantum result and the semiclassical result of the zeta function (the Riemann–Siegel lookalike formula equation (12)) for $N = 80$ (even parity). The vertical lines indicate the positions of quantum exact eigenvalues. The red curve (solid curve) is the quantum exact zeta function $Z_B(\omega)$. The green curve (dotted curve) is the semiclassical zeta function $Z_B^{(\mathrm{sc,RS})}(\omega)$, namely equation (12). $Z_B^{(\mathrm{sc,RS})}(\omega)$ does not cross the zero axis at several points where the quantum exact eigenvalues exist. These phenomena have already been observed (Saraceno and Voros 1994). These phenomena manifest that the semiclassical eigenvalues obtained from $Z_B^{(\mathrm{sc,RS})}(\omega)$ are in the complex domain.

In figure 2, we depict the quantum exact eigenvalues and the semiclassical eigenvalues which are obtained from $Z_B^{(\mathrm{sc,RS})}(\omega)$, equation (12). As we can see, some of the semiclassical eigenvalues by equation (12) are away from the quantum exact one, namely some eigenvalues are complex-valued. Note that one semiclassical eigenvalue is outside the frame of the figure, namely far from the unit circle. Thus the coincidence between those is rather bad.

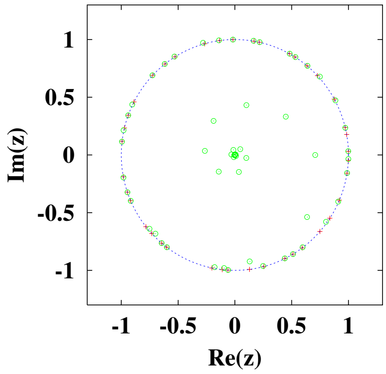

Figure 3. Semiclassical eigenvalues in the complex plane (semiclassical quantization of the baker's map summing up all periodic orbit contributions): $N = 1/(2\pi\hbar) = 80$. The data are for even parity. Circles indicates the positions of the semiclassical eigenvalues which are obtained by summing up all periodic orbit contributions. Crosses indicates the positions of the quantum exact eigenvalues. The coincidence between the quantum exact results and the semiclassical results is quite nice, although some semiclassical eigenvalues are complex-valued and an infinite number of irrelevant semiclassical eigenvalues exist inside the unit circle.

Next, in figure 3, we depict the quantum exact eigenvalues and the semiclassical eigenvalues which are obtained as the eigenvalues of $\mathcal{L}_B$. In this case, the coincidence is good, although the semiclassical eigenvalues inside the unit circle are irrelevant. The number of irrelevant semiclassical eigenvalues is infinite, because the dimension of the operator $\mathcal{L}_B$ is infinite. If we increase N (i.e., the semiclassical limit), the coincidence becomes better, namely $\left|\omega_n - \omega_n^{(\mathrm{sc})}\right| = O(\hbar^{1/2}) = O(N^{-1/2})$. The comparison between figures 2 and 3 shows that the Riemann–Siegel lookalike formula $Z_B^{(\mathrm{sc,RS})}(\omega)$ does not work well and the semiclassical quantization including all periodic orbit contributions by $Z_B^{(\mathrm{sc})}(\omega)$ is the best result as the lowest order asymptotics w.r.t. $\hbar$. The method B for the quantum baker's map is very similar to the resummation method by Gutzwiller which uses the Ising spin variables (Gutzwiller 1982). Actually Gutzwiller's resummation method can be applied to the quantum baker's map (Sano 2000).

4. Quantized sawtooth map

The classical sawtooth map is defined as follows. Its Hamiltonian is given by

$$H = \frac{y^2}{2} - K\frac{x^2}{2}\sum_{n=-\infty}^{\infty}\delta(t-n), \tag{25}$$

where $x, y \in \mathcal{D}$, $\mathcal{D} = \left[-\frac{1}{2}, \frac{1}{2}\right)$. The map is

$$x_{t+1} = x_t + y_{t+1} \bmod 1 \text{ in } \mathcal{D}, \qquad y_{t+1} = y_t + Kx_t \bmod 1 \text{ in } \mathcal{D}. \tag{26}$$

If we use the winding numbers around the torus for one step, the map is

$$x_{t+1} = x_t + y_{t+1} - w_x^{(t)} \equiv u(x_t, y_t), \qquad y_{t+1} = y_t + Kx_t - w_y^{(t)} \equiv v(x_t, y_t), \tag{27}$$

where $w_x^{(t)}$ and $w_y^{(t)}$ are the winding numbers and integers. If K is the integer, the sawtooth map becomes the Arnold cat map (Hannay and Berry 1980, Keating 1991). The sawtooth map is hyperbolic for $K < -4$, $K > 0$ and elliptic for $-4 \leqslant K \leqslant 0$. We define

$\lambda \equiv (K+2+\sqrt{K(K+4)})/2$. The Lyapunov exponent of this map is $\gamma = \ln\lambda$. For our convenience, we consider only the case of $K > 0$ (hyperbolic).

For the quantum system, the Floquet operator is given by

$$\hat{U}_S = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{\hat{y}^2}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}K\frac{\hat{x}^2}{2}}. \tag{28}$$

For this Floquet operator, the spectral determinant for the quantized sawtooth map is defined by

$$Z_S(\omega) = C_S(\omega)\,\det(1 - z^{-1}\hat{U}_S), \tag{29}$$

where S stands for the sawtooth map. The behaviour of the zeta function $Z_S^{(\mathrm{sc,RS})}(\omega)$ using the Riemann–Siegel lookalike formula has been reported by Sano (1996). As observed in figure 1 for the quantized baker's map, for some parameter value K except the cases that the semiclassical result is exact (K is an integer with $K < -4$, $K > 0$), $Z_S^{(\mathrm{sc,RS})}(\omega)$ does not cross the zero axis near some quantum exact eigenvalues. See figure 6 in Sano (1996). Thus similar to the quantized baker's map, the Riemann–Siegel lookalike formula does not produce accurate semiclassical eigenvalues.

Now we define the weighted Perron–Frobenius operator $\mathcal{L}_S$ for the sawtooth map,

$$\mathcal{L}_S(X, Y, x, y; N) = \mathrm{e}^{\frac{\gamma}{2}} \exp\left[\frac{\mathrm{i}}{\hbar}\phi\right]\delta(X - u(x, y))\delta(Y - v(x, y)), \tag{30}$$

where

$$\phi = \frac{y^2}{2} - \frac{Kx^2}{2} + (Y - y)x. \tag{31}$$

See the appendix for the semiclassical approximation of the trace $\mathrm{Tr}\big(\hat{U}_S^T\big)$. In the appendix, it is shown that the Maslov index for the periodic orbit p (period $T = T_p r$) is $\nu_p r = T$ for $K > 0$. Thus, $\mathrm{i}^T\,\mathrm{e}^{-\frac{\pi\nu_p r}{2}} = 1$. This implies that the phase contribution from each periodic orbit comes only from the action for $K > 0$. Therefore, in equation (30), the contribution from the Maslov index does not appear. By equation (4), it is easily shown that

$$\mathrm{Tr}\big(\hat{U}_S^n\big) \simeq \mathrm{Tr}^{(\mathrm{sc})}\big(\hat{U}_S^n\big) = (1 - \mathrm{e}^{-\gamma n})\,\mathrm{Tr}\big(\mathcal{L}_S^n\big), \tag{32}$$

because

$$\begin{aligned}
\frac{\mathrm{e}^{\gamma n/2}(1 - \mathrm{e}^{-\gamma n})}{\big|\det\big(M_p^n - I\big)\big|} &= \frac{\mathrm{e}^{\gamma n/2} - \mathrm{e}^{-\gamma n/2}}{|(\mathrm{e}^{\gamma n} - 1)(\mathrm{e}^{-\gamma n} - 1)|} \\
&= \frac{1}{\mathrm{e}^{\gamma n/2} - \mathrm{e}^{-\gamma n/2}} \\
&= \frac{1}{2\sinh(\gamma n/2)} \\
&= \frac{1}{\big|\det\big(M_p^n - I\big)\big|^{1/2}}.
\end{aligned} \tag{33}$$

Using equation (32), we have

$$\begin{aligned}
\det(1 - z^{-1}\hat{U}_S) &= \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Tr}(\hat{U}^n)z^{-n}\right] \\
&\simeq \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Tr}^{(\mathrm{sc})}(\hat{U}^n)z^{-n}\right]
\end{aligned}$$

$$= \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}(1-\mathrm{e}^{-\gamma n})\,\mathrm{Tr}\big(\mathcal{L}_S^n\big)z^{-n}\right]$$

$$= \frac{\det(1-z^{-1}\mathcal{L}_S)}{\det(1-\mathrm{e}^{-\gamma}z^{-1}\mathcal{L}_S)}. \tag{34}$$

Thus, the zeta function $Z_S(\omega)$ is approximated as

$$Z_S(\omega) \simeq Z_S^{(\mathrm{sc})}(\omega) = C_S(\omega)\frac{\det(1-z^{-1}\mathcal{L}_S)}{\det(1-\mathrm{e}^{-\gamma}z^{-1}\mathcal{L}_S)}. \tag{35}$$

This is the central result of this section. Unfortunately when we represent the weighted Perron–Frobenius operator in the matrix representation using the Fourier basis set, the matrix elements are written as the integration of wildly oscillating function. Thus numerical evaluation of the matrix elements is hard. But from the above formula, it is easily understood that some eigenvalues of $\mathcal{L}_S$ (zeros of $Z_S^{(\mathrm{sc})}(\omega)$) give the semiclassical eigenvalues for $\hat{U}_S$. In addition, $Z_S^{(\mathrm{sc})}(\omega)$ has not only zeros but also poles. For the positive integer K, the map becomes the Arnold cat map and the semiclassical approximation is exact. Thus for the positive integer K, $Z_S(\omega) = Z_S^{(\mathrm{sc})}(\omega)$ and $Z_S(\omega)$ is the finite product. Therefore, the zeros of $\det(1-z^{-1}\mathcal{L}_S)$ except the eigenvalues of $\hat{U}_S$ cancel all poles of $1/\det(1-\mathrm{e}^{-\gamma}z^{-1}\mathcal{L}_S)$. Although the evaluation of the matrix elements for $\mathcal{L}_S$ is difficult, the Pollicott–Ruelle resonances of the classical sawtooth map, i.e., the eigenvalues of the operator $\mathcal{L}_{S,cl} = \delta(X-u(x,y))\delta(Y-v(x,y))$, were numerically calculated (Sano 2002).

5. Summary

In conclusion, we have demonstrated that the semiclassical quantization based on the Riemann–Siegel lookalike formula for the maps on torus (i.e., the baker's map and the sawtooth map) fails to yield its eigenvalues. We have shown that for the baker's map, the alternative semiclassical quantization scheme including all periodic orbit contribution (i.e., the weighted Perron–Frobenius operator) does yield its eigenvalues with reasonable accuracy, although they are complex-valued. Our demonstration suggests that for the quantized maps on torus, the bootstrap effect in the Riemann–Siegel lookalike formula does not work well and, after all, all periodic orbit contributions are needed for accurate semiclassical quantization. At present, we do not know whether this behaviour is peculiar for the quantized map on a torus or not, and whether it is also observed for general autonomous systems or not.

There is one critique of our method (method B). The two example maps (i.e., the baker's map and the sawtooth map) are linear. We used the linearity of the map to construct the weighted Perron–Frobenius operator. Precisely, we used the uniform hyperbolicity, that is, the Lyapunov exponent of each trajectory is the same. For nonlinear maps, e.g., the kicked rotator or the kicked top, the construction of the weighted Perron–Frobenius operator is, in general, difficult. This is a weak point of our method.

Appendix

In this appendix, we show that for $K > 0$, when we consider the semiclassical trace $\mathrm{Tr}^{(\mathrm{sc})}\big(\hat{U}_S^T\big)$, the Maslov index ν_p for all periodic orbits is $\nu_p r = T$. The Hamiltonian of the quantum sawtooth map is given by

$$\hat{H} = f(\hat{y}) + g(\hat{x})\sum_{n=-\infty}^{\infty}\delta(t-n), \tag{A.1}$$

where

$$f(\hat{y}) = \frac{\hat{y}^2}{2}, \qquad g(\hat{x}) = -\frac{K\hat{x}^2}{2}. \tag{A.2}$$

The matrix elements of the Floquet operator $\hat{U}_S$ are given by

$$\langle n|\hat{U}_S|m\rangle = (-1)^{n-m}\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{\sqrt{N}}\exp\left[\frac{\mathrm{i}\pi}{N}Km^2\right]\exp\left[\frac{\mathrm{i}\pi}{N}(n-m)^2\right], \tag{A.3}$$

where $-\frac{N}{2} \leqslant n, m \leqslant \frac{N}{2} - 1$. We have used the Gauss sum

$$\sum_{n=0}^{N-1}\exp\left[\frac{\mathrm{i}\pi}{N}n^2\right] = \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\sqrt{N}. \tag{A.4}$$

The trace of the Floquet operator becomes

$$\mathrm{Tr}\big(\hat{U}_S^T\big) = \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi T}{4}}}{N^{\frac{T}{2}}}\sum_{l_1,\ldots,l_T=-\frac{N}{2}}^{\frac{N}{2}-1}\exp\left[\frac{\mathrm{i}\pi}{N}\sum_{j=1}^{T}\big\{(K+2)l_j^2 - 2l_jl_{j+1}\big\}\right]. \tag{A.5}$$

The semiclassical approximation of the trace is given by (Lebœuf and Mouchet 1994)

$$\mathrm{Tr}^{(\mathrm{sc})}\big(\hat{U}_S^T\big) = \mathrm{i}^T\sum_{p,T_pr=T}\frac{T_p}{\sqrt{\big|\det\big(M_p^T - I\big)\big|}}\exp\left[\frac{\mathrm{i}}{\hbar}S_pr - \frac{\mathrm{i}\pi\nu_pr}{2}\right], \tag{A.6}$$

with

$$S_pr = \sum_{i=1}^{T}\big\{-(f(y_i)+g(x_i)) + y_{i+1}(x_{i+1}-x_i) - w_y^{(i)}x_i + w_x^{(i-1)}y_i\big\}. \tag{A.7}$$

Our aim is to show $\nu_pr = T$. Using the Poisson summation formula, equation (A.5) is rewritten as

$$\mathrm{Tr}\big(\hat{U}_S^T\big) = \mathrm{e}^{-\frac{\mathrm{i}\pi T}{4}}N^{\frac{T}{2}}\sum_{l_1,\ldots,l_T=-\infty}^{\infty}\int_{\mathcal{D}^T}\mathrm{d}\mathbf{x}\,\exp\left[\frac{\mathrm{i}}{\hbar}\phi(\mathbf{l},\mathbf{x})\right], \tag{A.8}$$

where $\hbar = \frac{1}{2\pi N}$, $x_i \in \mathcal{D}$ and $\mathbf{l} = (l_1, l_2, \ldots, l_T)$, $l_i \in \mathbb{Z}$. The function $\phi(\mathbf{l},\mathbf{x})$ is given by

$$\phi(\mathbf{l},\mathbf{x}) = \mathbf{l}\cdot\mathbf{x} + \tfrac{1}{2}\mathbf{x}^{\mathbf{T}}\cdot\mathbf{A}\cdot\mathbf{x}, \tag{A.9}$$

where the matrix $\mathbf{A}$ is the $T \times T$-symmetric matrix,

$$\mathbf{A} = \begin{pmatrix} K+2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & K+2 & -1 & & & 0 \\ 0 & -1 & K+2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & -1 & 0 \\ 0 & & & -1 & K+2 & -1 \\ -1 & 0 & \ldots & 0 & -1 & K+2 \end{pmatrix}. \tag{A.10}$$

The integrals in equation (A.8) are evaluated by the stationary phase approximation. We only pick up the integrals that the stationary phase point is inside of $\mathcal{D}^T$, namely neglecting the boundary contribution of the integral. Hence for the integral picked up, we replace the integration domain $\mathcal{D}^T$ by $[-\infty, \infty]^T$. This is the lowest order approximation by asymptotics.

Here we consider an integral from one stationary contribution specified by $\mathbf{l}^*$. This contribution is the periodic orbit p with the repetition r.

$$I = \mathrm{e}^{-\frac{\mathrm{i}\pi T}{4}} N^{\frac{T}{2}} \int_{[-\infty,\infty]^T} \mathrm{d}\mathbf{x} \, \exp\left[\frac{\mathrm{i}}{\hbar}\left(\mathbf{l}^* \cdot \mathbf{x} + \frac{1}{2}\mathbf{x}^{\mathbf{T}} \cdot \mathbf{A} \cdot \mathbf{x}\right)\right]. \tag{A.11}$$

Carrying out the Gaussian integral, we have

$$I = \sqrt{\frac{|\det(\mathbf{A})|}{\det(\mathbf{A})}} \sqrt{\frac{1}{|\det(\mathbf{A})|}} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_p r}, \tag{A.12}$$

where $|\det(\mathbf{A})| = |\det(M^T - I)|$ and

$$M = \begin{pmatrix} K+1 & 1 \\ K & 1 \end{pmatrix}. \tag{A.13}$$

By equation (A.12), the Maslov index ν_p is given by

$$\mathrm{i}^T \, \mathrm{e}^{-\frac{\mathrm{i}\pi\nu_p r}{2}} = \sqrt{\frac{|\det(\mathbf{A})|}{\det(\mathbf{A})}}. \tag{A.14}$$

It is easily shown that the eigenvalues of the matrix $\mathbf{A}$ are

$$\Lambda_i = K + 2 - 2\cos\left(\frac{2\pi \mathrm{i}}{T}\right), \tag{A.15}$$

$i = 1, 2, \ldots, T$. Therefore, $\Lambda_i > 0\,(i = 1, 2, \ldots, T)$ for $K > 0$. Thus, $\det(\mathbf{A}) > 0$. Then, $\nu_p r = T$.

References

Balazs N L and Voros A 1987 *Europhys. Lett.* **4** 1089
Balazs N L and Voros A 1989 *Ann. Phys.* **190** 1
Berry M V and Keating J P 1990 *J. Phys. A: Math. Gen.* **23** 4839
Berry M V and Keating J P 1992 *Proc. R. Soc.* A **437** 151
Dittes F-M, Doron E and Smilansky U 1994 *Phys. Rev.* E **49** R963
Gutzwiller M C 1971 *J. Math. Phys.* **12** 343
Gutzwiller M C 1982 *Physica* D **5** 183
Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
Hannay J H and Berry M V 1980 *Physica* D **1** 267
Keating J P 1991 *Nonlinearity* **4** 277
Keating J P 1991 *Nonlinearity* **4** 309
Keating J P 1992 *Proc. R. Soc.* A **436** 99
Keating J P 1994 *J. Phys. A: Math. Gen.* **27** 6605
Lebœuf P and Mouchet A 1994 *Phys. Rev. Lett.* **73** 1360
Sano M M 1996 *J. Phys. A: Math. Gen.* **29** 6087
Sano M M 2000 *Chaos* **10** 195
Sano M M 2002 *Phys. Rev.* E **66** 046211
Saraceno M 1990 *Ann. Phys.* **199** 37
Saraceno M and Voros A 1994 *Physica* D **79** 206
Smilansky U 1993 *Nucl. Phys.* A **560** 57
Smilansky U and Verdene B 2003 *J. Phys. A: Math. Gen.* **36** 3525
Tanner G 1999 *J. Phys. A: Math. Gen.* **32** 5071